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TWO-PHASE FLOW IN A VERTICAL PIPE AND THE PHENOMENON OF CHOKING: HOMOGENEOUS DIFFUSION MODEL—I

HOMOGENEOUS FLOW MODELS

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Abstract—The paper examines the topological structure of all possible solutions which can exist in flows through adiabatic constant-area ducts for which the homogeneous diffusion model has been assumed. The conservation equations are one-dimensional with the single space variable z, but gravity effects are included. The conservation equations are coupled with three equations of state: a pure substance, a perfect gas with constant specific heats, and a homogeneous two-phase system in thermodynamic equilibrium. The preferred state variables are pressure P, enthalpy h, and mass flux G^2.

The three conservation equations are first-order but nonlinear. They induce a family of solutions which are interpreted as curves in a four-dimensional phase space conceived as a union of three-dimensional spaces (P, h, G^2; z) with G^2 = const treated as a parameter. It is shown that all points in these spaces are regular, so that no singular solutions need to be considered. The existence and uniqueness theorem leads to the conclusion that through every point in phase space there passes one and only one solution-curve.

The set of differential equations, treated as a system of algebraic equations of each point of the phase space, determines the components of a rate-of-change vector which are obtained explicitly by Cramer's rule. This vector is tangent to the solution curve. Each solution curve turns downward in z at some specific elevation z^*, and this determines the condition for choking. Choking occurs always when the exit flow velocity at $L = z^*$ is equal to the local velocity of propagation of small plane disturbances of sufficiently large wavelength, that is when the flow rate G becomes equal to a specified, critical flow rate, G^*. (The possible dependence of the sonic velocity on frequency in a real flow is ignored, because it has not been allowed for in the equations of the model under study.) A criterion, analogous to the Mach number, which indicates the presence or absence of choking in a cross section is the ratio $K = G/G^{**}$ of the mass-flow rate G to the local critical mass flow rate, G^{**}, $K = 1$ denoting choking. The critical parameters depend only on the thermodynamic properties of the fluid and are independent of the gravitational acceleration and shearing stress at the wall.

The topological characteristics of the solutions allow us to study all flow patterns which can, and which cannot, occur in a pipe of given length L into which fluid is discharged through a rounded entrance from a stagnation reservoir and whose back-pressure is slowly lowered. The set of flow patterns is analogous to that which occurs with a perfect gas, except that the characteristic numerical values are different. They must be obtained by numerical integration and the influence of gravity must be allowed for.

The preceding conclusions are valid for all assumptions concerning the shearing stress at the wall which make it dependent on the state parameters only, but not on their derivatives with respect to z. However, the study is limited to upward flows for which the shearing stress at the wall and the gravitational acceleration are codirectional.

1. INTRODUCTION

The motivation for this paper is a desire to study the upward flow of geothermal brines through vertical wells and to determine the conditions under which they choke. As a first approximation the brine is assumed to consist of water substance which may occur in one or two phases. The presence of gaseous or solid solutes is ignored for the present, and the flow is treated as adiabatic.

Except for the added effects of considerable changes in potential energy—$g\,dz$ per unit mass, with z denoting the vertical space coordinate—the analysis to be presented here can be of use in several other applications, notably in the study of certain elements of nuclear reactors, the piping employed for interstage reheating or feed-water heating in wet steam turbines, as well as in the long ducts of geothermal installations.

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At this stage, the analysis will be confined to homogeneous flow models, to steady flows, and to pipes of constant cross section. The method of analysis is a natural extension of the earlier work of one of the authors with his then collaborators (Kestin & Oppenheim 1948; Kestin & Zaremba 1952–54).

The study of choking of two-phase flows in pipes has been undertaken by many research workers, notably by Bouré *et al.* (1976), Hsu & Graham (1976), Wallis (1969) and others. Summary reviews have recently been published by Giot (in Delhaye, Giot & Riethmuller 1981), Bouré (in Ginoux 1978) and Wallis (1980). However, no unambiguous criterion for the occurrence of choking has been formulated.

2. METHOD

The analysis consists of several distinctive steps.

First, we write the three conservation equations in terms of space averages, leaving in them a single space variable, the axial distance z. All quantities are assumed to be time averages and turbulent fluctuations are not introduced explicitly. This is the so-called one-dimensional or "engineering" approximation.

Secondly, we set up a flow model for two-phase flow. In the present study, we restrict ourselves to the *homogeneous diffusion model* (Ginoux 1978; Wallis 1969). This is done for simplicity†.

The model assumes a constant pressure and temperature in every cross section as well as thermodynamic equilibrium between the phases. Surface-tension effects are neglected and infinite evaporation and condensation rates are stipulated.

Thirdly, we assume an equation of state. In this respect it is helpful to analyze in parallel the case of general single-phase flow and the flow of a perfect gas with constant specific heats as well as to mention the case when a two-phase flow is choked after all of the liquid has evaporated.

Fourthly, we discuss in detail the set of possible solutions admitted by the three coupled ordinary nonlinear differential equations provided by the flow model. The analysis is geometrical and is conducted in a three-dimensional phase space chosen for its simplicity. The same mathematical method can be used for the case of the more complex models, including, *mutatis mutandis*, the two-fluid model.

The essential novelty of the method, as applied to problems of two-phase flow, is the full use of the existence and uniqueness theorem applicable to the set of three equations under discussion (Kestin & Zaremba 1952; Birkhoff & Rota 1969; Kaplan 1958) and the study of the topological nature of the set of solutions. Except for the case of a perfect gas with constant specific heats, the actual solution must be numerical.

3. THE PROBLEM IN THE PHYSICAL SPACE

The nature of the problem in the physical space is illustrated in figure 1. We are given a vertical adiabatic pipe of constant cross-sectional area $A = \pi D^2/4$ and length L through which there flows upwards and in steady-state a fluid which may evaporate and condense. The fluid is acted upon by the terrestrial gravitational acceleration, g. We wish to determine all possible flows in terms of the back-pressure P_a maintained outside $z_e = L$, and to identify the subset of initial conditions which may occur at the entrance to the pipe at $z_1 = 0$. We regard the pipe as a continuous thermodynamic system whose "state" is described by *three* quantities: two "proper" thermodynamic properties (we shall favor the pressure P and enthalpy h) and one

†In order to concentrate on the problem at hand we do not attempt here to pinpoint the desirable nomenclature for the different flow models proposed in the literature. The model employed is clear from the equations if not from the term. The only matter that may arouse discussion is whether the model used here can be applied, by way of acceptable approximation, to flows with slip, in which case the closure equation for the shearing stress would contain the velocity different between vapor and fluid, $w_G - w_L$, and a second closure equation would be needed.

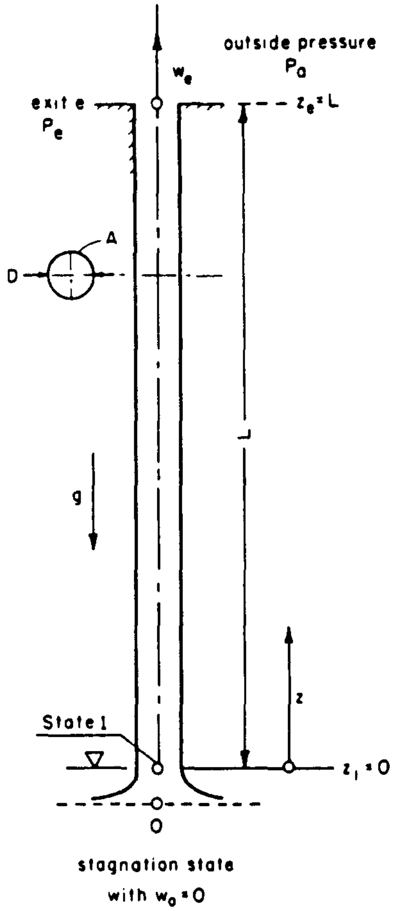

Figure 1. The physical problem. Note that P_e need not be equal to P_a, but $P_e \geq P_a$.

hydrodynamic property (we shall favor the mass-flow rate $G = \dot{m}/A = \rho w$, where w is the velocity and ρ is the average density). The state of our continuous system is thus described by the three functions ("fields"):

$$P(z), h(z) \quad \text{and} \quad G(z),$$

the latter being constant for a given flow, but varying as the boundary conditions outside the pipe are varied. In particular, we shall be interested in the relation of the exit pressure, P_e, to the external pressure P_a, to which it may, but need not, be equal, remembering however, that $P_e \geq P_a$ for outflow. We are also interested in the variation of the mass flow rate $\dot{m} = GA$ with P_e.

We realize at this point that there exists no direct mechanism which would allow us to impose on the pipe both the thermodynamic conditions at inlet $z_1 = 0$ as well as the flow rate G. The latter is also affected by the flow resistance, that is by the pressure drop $P_e - P_1$ suffered by the pipe. We can think in terms of a fictitious stagnation reservoir in which there is maintained a fixed state, subscript 0, and from which the fluid is accelerated isentropically from $w_0 = 0$ to w_1 as the back-pressure is varied. In analogy with the familiar case of a gas, we can foresee, and analysis will confirm, that for a fixed stagnation state the mass flux, G, treated as a function of the back-pressure P_a will reach a maximum for a particular value P^* of P_e. This is the process of choking which occurs for a specified set of conditions at inlet $z_1 = 0$. It is these conditions that remain to be discovered in the course of our analysis.

4. THE CONSERVATION EQUATIONS

The homogeneous diffusion model is specified by the following three conservation equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}(A\rho w) = 0; \qquad [1]$$

$$\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}(\rho w^2) + g\rho + \frac{C}{A}\tau_w = 0; \qquad [2]$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(h + \frac{w^2}{2} + gz\right) = 0. \qquad [3]$$

This set of equations is familiar from elementary gas dynamics (Liepmann & Roshko 1957; Shapiro 1953) for single-phase flows, with C denoting the circumference of the cross section A. The three coupled, first-order non-linear differential equations contain five "state" variables

$$\rho(z), w(z), P(z), h(z), \tau_w(z).$$

The system of equations is closed by adding an equation of state {we shall favor equations of state of the form

$$\rho = \rho(P, h), \qquad [4]$$

valid for single as well as two-phase flow} and by a *closure equation*† for shearing stress, τ_w. Regarding the latter we merely stipulate that it is a relation between state variables

$$F(\rho, w, P, h, \tau_w) = 0, \quad (\tau_w > 0), \qquad [5]$$

to the exclusion of their derivatives with respect to z. As indicated in an earlier footnote we leave open the question of the suitability of this set of equations to deal, approximately, with flow in the presence of relative velocity. The subsequent discussion remains valid in the presence of relative velocity as long as [5] contains a measure of it, but not its z-derivative, and as long as the additional closure equation is of a form analogous to [5].

In the case of two-phase flow, the velocity w is the barycentric velocity defined by

$$\rho w = w_L\rho_L(1-\alpha) + w_G\rho_G\alpha, \qquad [6]$$

where α is the void fraction, and $\rho_L(P)$, $\rho_G(P)$ are known functions of (saturation) pressure. The extensive specific properties Φ, (such as enthalpy h, entropy s, etc.), all expressed per unit mass, obey the general equation

$$\Phi = (1-x)\Phi_L + x\Phi_G \qquad [7]$$

where $\Phi_L(P)$, $\Phi_G(P)$ are known functions and x is the dryness fraction (equal to quality only in the absence of relative velocity). The void fraction, α, and dryness fraction, x, are interchangeable owing to the existence of the equivalence relation

$$\alpha = x\rho/\rho_G = \frac{1}{\rho_G/x\rho_L - \rho_G/\rho_L + 1}. \qquad [8]$$

†We prefer to use this term borrowed from the theory of turbulence rather than the terms "constitutive law" or "external constitutive law" recently proposed. The term used here appears to be semantically more appropriate.

The mass flux $\dot{m}_G = -\dot{m}_L$ which results from a change of phase is governed by the partial equation of mass conservation

$$\dot{m}_G = -\dot{m}_L = \frac{\mathrm{d}}{\mathrm{d}z}(A\alpha\rho_G w) \qquad [9]$$

which need not enter our discussion because the mass flux in it is uniquely determined once the main system of three equations has been solved.

5. WORKING EQUATIONS

We can considerably simplify the analysis by choosing the following independent variables:

$$G^2(z), P(z), h(z). \qquad [10]$$

We stipulate $A = \pi D^2/4 = \text{const}$ and obtain

$$\frac{\mathrm{d}G^2}{\mathrm{d}z} = 0; \qquad [11a]$$

$$\left[1 - \frac{G^2C_1}{\rho^2}\right]\frac{\mathrm{d}P}{\mathrm{d}z} - \frac{G^2C_2}{\rho^2}\frac{\mathrm{d}h}{\mathrm{d}z} = -\frac{4\tau_w}{D} - \rho g; \qquad [11b]$$

$$-\frac{G^2C_1}{\rho^3}\frac{\mathrm{d}P}{\mathrm{d}z} + \left[1 - \frac{G^2C_2}{\rho^3}\right]\frac{\mathrm{d}h}{\mathrm{d}z} = -g. \qquad [11c]$$

Here

$$C_1 = \left(\frac{\partial\rho}{\partial P}\right)_h \text{(always} > 0\text{)}; \quad C_2 = \left(\frac{\partial\rho}{\partial h}\right)_P \text{(always} < 0\text{)}, \qquad [12a, b]$$

for single as well as two-phase flows.

As far as the equation of state, $\rho = \rho(P, h)$, is concerned, we have

$$\rho = c_p P/Rh \qquad [13a]$$

for a perfect gas, and

$$\frac{1}{\rho} = U(P)h + V(P) \qquad [13b]$$

for a two-phase system in thermodynamic equilibrium. Here

$$U(P) = (v_G - v_L)/(h_G - h_L) \quad \text{and} \quad V(P) = v_L - h_L(v_G - v_L)/(h_G - h_L) \qquad [14a, b]$$

are assumed to be known, say, from *ad hoc* fits to thermodynamic tables.

6. THE MATHEMATICAL PROBLEM

The three conservation equations [11a–c] together with the substitutions [4] and [5] constitute a set of three coupled ordinary nonlinear first-order differential equations for the three variables G^2, P, h, conceived as functions of the single variable z. From the physical point of view the three variables G^2, P, h are independent and together determine the *state* in a cross section. Treated as functions of the single independent *space* variable z, that is from the mathematical point of view, they are the dependent functions of the problem: $G^2(z)$, $P(z)$, and $h(z)$. This set of three parametric equations traces a single curve in the four-dimensional phase space G^2, P, h; z for each particular flow through a vertical pipe.

Regardless of the equation of state, the system of differential equations can be contracted to the matrix equation

$$A_{ij}(\sigma)\frac{\mathrm{d}\sigma_i}{\mathrm{d}z} = B_j(\sigma) \quad (i, j = 1, 2, 3) \tag{15}$$

in which the row vector $\mathrm{d}\sigma_i/\mathrm{d}z$ has components $\mathrm{d}G^2/\mathrm{d}z$, $\mathrm{d}P/\mathrm{d}z$ and $\mathrm{d}h/\mathrm{d}z$. The matrix A_{ij} possesses the simple structure

$$A_{ij} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}. \tag{16}$$

The elements A_{22}, A_{23}, A_{32}, A_{33} are listed explicitly in table 1 for the specialized cases of a perfect gas and a two-phase mixture in equilibrium. The components of the column vector B_j

$$B_j \equiv \begin{bmatrix} 0 \\ B_2 \\ B_3 \end{bmatrix} \tag{17}$$

are listed in table 2. Their general form is recorded in [11a–c].

The direct method of dealing with [15] is to set up a computer program and to solve the equations by a step-by-step procedure starting with a given set of initial conditions. At each step we would solve a system of simultaneous, linear *algebraic* equations in which the elements

Table 1. The elements of matrix [16]

	Perfect gas	Equilibrium mixture
A_{22}	$1 - \frac{G^2Rh}{c_pP^2}$	$1 + G^2\left(\frac{dU}{dP}h + \frac{dV}{dP}\right)$
A_{23}	$\frac{G^2R}{c_pP}$	G^2U
A_{32}	$-\frac{G^2R^2h^2}{c_p^2P^3}$	$G^2(Uh + V)\left(\frac{dU}{dP}h + \frac{dV}{dP}\right)$
A_{33}	$1 + \frac{G^2R^2h}{c_p^2P^2}$	$1 + G^2U(Uh + V)$

Table 2. The elements of vector [17]

	Perfect gas	Equilibrium mixture
B_2	$-\frac{c_pP}{Rh}g - \frac{4\tau_w}{D}$	$-\frac{4\tau_w}{D} - \frac{g}{Uh + V}$
B_3	$-g$	$-g$

of the matrix A_{ij} and of the column vector B_j become given numbers. The initial conditions would determine a value of G^2 which stays constant in a given flow, and each step in the calculation of $\mathrm{d}P/\mathrm{d}z$ and $\mathrm{d}h/\mathrm{d}z$ would require the solution of two simultaneous algebraic equations, say, (but not necessarily), by Cramer's rule. In the latter case we would obtain

$$\frac{\mathrm{d}P}{\mathrm{d}z} = \frac{N_P}{\Delta} \quad \text{and} \quad \frac{\mathrm{d}h}{\mathrm{d}z} = \frac{N_h}{\Delta}, \qquad [18a, b]$$

where

$$\Delta \equiv \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} \qquad [19]$$

is the determinant. Explicit expressions for the three quantities: N_P, N_h, and Δ are given in table 3. Such a direct procedure has several drawbacks. First, it is not certain that there exists a solution for the assumed initial conditions. Secondly, if a step involving $\Delta = 0$ were encountered, the computer program would fail to calculate the step. Thirdly, the onset of choking could only be established by a search procedure for a maximum in G with varying initial conditions and complex matching to the back-pressure, engaging the computer in a good deal of groping iteration.

It is sometimes supposed that the onset of choking can be deduced from the character of the system of algebraic equations, and it is asserted that $\Delta = 0$ represents a necessary critical flow criterion (Bouré in Ginoux 1978, p. 185). Although the above statement will prove to be correct in our case, it is not clear that it possesses general validity. The analysis given in the succeeding sections of this paper will present an unambiguous method of identifying the onset of choking conditions.

At this stage we can only assert that the two algebraic equations may occur in three versions†:

(1) $\Delta \neq 0$. In this case, regardless of the sign of N_p and N_h, the equations possess a unique solution and the numerical solution (such as by the Runge-Kutta procedure) can progress by one step.

(2) $\Delta = 0$. In this case either (a) $N_p \neq 0$ and $N_h \neq 0$, or (b) $N_p = N_h = 0$. The vanishing of only one of N_h, N_p without the other is not possible.

(2a) The two algebraic equations are inconsistent.‡ Without a proper analytic investigation of the neighborhood of this state with $\Delta \to 0$ instead of $\Delta = 0$, it is not possible to draw any conclusions regarding the behavior of our flow.

(2b) The two algebraic equations are compatible, but one is a multiple of the other, and an infinity of solutions for $\mathrm{d}h/\mathrm{d}z$ and $\mathrm{d}P/\mathrm{d}z$ is possible. Again, without a proper analysis of the neighborhood of this state, no further conclusions can be drawn. The explicit forms displayed in table 3 demonstrate that this case cannot occur in our present problem, though it may appear in downward flow through a channel of constant cross section or in channels with varying cross sections. A brief remark about this case will be made in section 7.

The theory of coupled ordinary differential equations, based on a simple extension of the method of isoclines to a multidimensional space, and interpreted geometrically, will enable us to obtain a clear topological description of all possible solutions to our problem thus laying the foundation for more efficient programming.

†The occurrence of homogeneous equations is excluded by the physics of our problem as seen from [11a–c] and table 2. However, an elementary search for the velocity of propagation of a plane disturbance would lead us to solving the three related homogeneous equations and to the conclusion that the propagation velocity is implied in the equation $\Delta = 0$.

‡In a manner of speaking, we can say that the two straight lines in $\mathrm{d}h/\mathrm{d}z$, $\mathrm{d}P/\mathrm{d}z$ represented by the two linear algebraic equations for them "intersect at infinity", suggesting that $\Delta \to 0$ implies $\mathrm{d}h/\mathrm{d}z \to \infty$ and $\mathrm{d}P/\mathrm{d}z \to \infty$.

Table 3. Explicit expressions for the components N_P, N_h, Δ

	General	Perfect gas	Equilibrium mixture
N_p	$= -\dfrac{4\tau_w(\rho^3 - C_2G^2) + Dg\rho^4}{D\rho^3}$	$-\dfrac{4\tau_w}{D}\left[1 + \dfrac{G^2(\gamma-1)^2h}{\gamma^2P^2}\right] - \dfrac{\gamma P_\gamma}{(\gamma-1)h}$ $N_p < 0$ for all states	$-\dfrac{4}{D}\tau_w - \dfrac{g}{Uh + V} - \dfrac{4}{D}\tau_w G^2(Uh+V)U$
N_h	$= -\dfrac{4\tau_w G^2C_1 + D\rho^3 g}{D^3}$	$-\dfrac{4\tau_w}{D}\dfrac{G^2h^2(\gamma-1)^2}{\gamma^2P^3} - g$ $N_h < 0$ for all states	$\dfrac{4}{D}G^2\tau_w\,(Uh + V)\left[\dfrac{dU}{dP}h + \dfrac{dV}{dP}\right] - g$
Δ	$= 1 - \dfrac{G^2}{\rho^2}C_1 - \dfrac{G^2}{\rho^3}C_2$	$1 - \dfrac{\gamma-1}{\gamma^2}\dfrac{G^2h}{P^2}$ $\Delta = 0$ for $G^{*2} = \rho^2a^2$ or	$1 + G^2\left(\dfrac{dU}{dP}h + \dfrac{dV}{dP}\right) + G^2U(Uh + V)$
G^{*2}	$= \dfrac{\rho^3}{C_1\rho + C_2}$	$\dfrac{\gamma^2}{\gamma-1}\dfrac{P^2}{h}$	$\left\{-\left(\dfrac{dU}{dP}h + \dfrac{dV}{dP}\right) - (Uh + V)U\right\}^{-1}$

There exists no method to deal with [15] in all its generality. Here we have encountered a very simple version of it, and this enables us to carry out a complete analysis. More complex cases can and will be discussed in later publications, but the gist of the approach should be clear from the present analysis.

7. THE TOPOLOGY OF THE PHASE SPACE

The property [11a] allows us to study the topology of the four-dimensional space (G^2, P, h, z) in terms of the set of subspaces (P, h, z), referred to as Γ, with $G^2 = \text{const}$ treated as a variable parameter. Since $0 < P < P_0$ and $0 < h < h_0$, we are only interested in the portion of Γ contained within these bounds. A point, M, in Γ represents a state in a cross section, and the flow through a pipe is represented by a curve, denoted by m in figure 2. This is a Fanno line well-known from elementary studies.†

Equations [15] determines a vector which at any point M is tangent to m. The projection $\theta = d\sigma_\alpha/dz$ ($\alpha = 2, 3$) is tangent to m in Γ. The components of θ, given by [18a, b] and listed explicitly in table 3, are: N_p in the direction P, N_h in the direction h, and Δ in the direction z.

The structure of the set of solutions m is determined by the structure of the vector field θ in Γ. This structure turns out to be very simple to sketch by noting several of its algebraic properties. The components N_P, N_h, Δ do not contain z explicitly and depend only on the thermodynamic properties P, h. Moreover, N_P and N_h are always negative and only Δ may change sign. This occurs when a particular combination P^*, h^* of P, h satisfies the relation

$$\Delta(P^*, h^*; G) = 0. \qquad [20]$$

The solution of [20] in the form

$$G = G^*(P^*, h^*) \qquad [21]$$

determines a line of critical states in Γ. As a final point we note that $|N_P|$ and $|N_h|$ increase in the negative directions of P and h, respectively.

Isoclines, i.e. lines along which the vector θ has a constant value, are straight and parallel to

†We restrict the discussion to upward flows.

z, figure 3. The isoclines with $\Delta = 0$ form a cylinder δ which divides Γ into two parts. In part Γ_1 with $\Delta > 0$ the vectors θ force m to ascend, whereas in Γ_2 with $\Delta < 0$, curves m descend, figure 4. In any plane z = const, the vector θ is normal to z along the intersection with $\Delta = 0$, and lies in that plane. Curves m turn at such points and exhibit maxima in z, denoted by z^*.

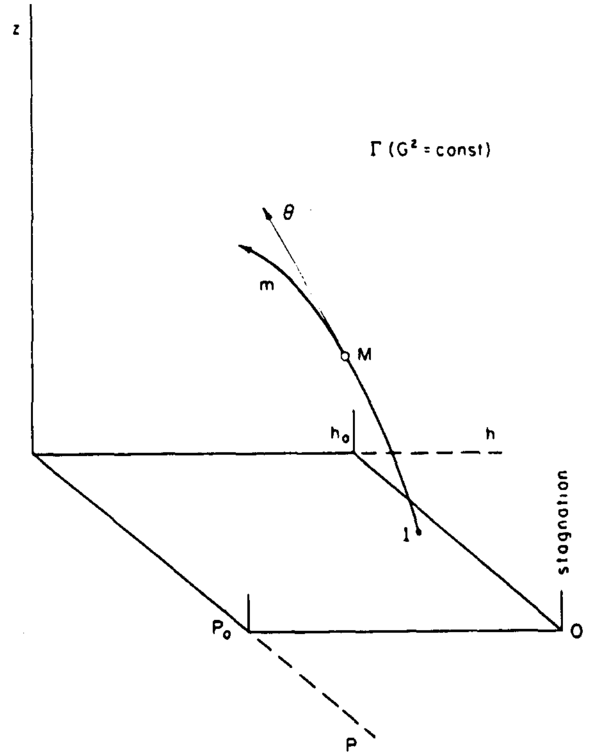

Figure 2. Fanno line in the subspace (h, P, z) denoted $\Gamma(G^2 = \text{const})$.

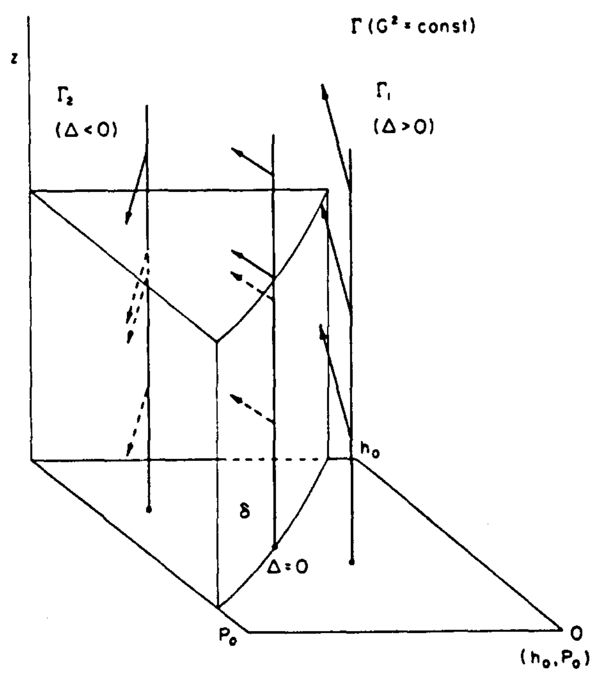

Figure 3. Isoclines in Γ-space.

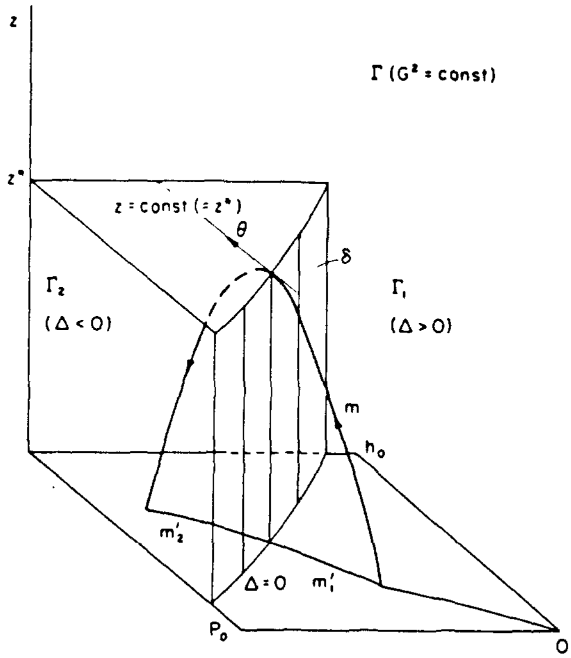

Figure 4. The turning of a solution.

We can complete the topological sketch of curves m by recalling the *local existence and uniqueness theorem* (see for example, Kaplan 1958; Ince 1956; Goursat 1959; Coddington & Levinson 1955). Since N_P, N_h are of one sign, there exist no points for which

$$N_P = N_h = \Delta = 0 \qquad [22]$$

simultaneously. Interpreted in terms of the theory of ordinary differential equations, this proves that Γ does not contain *singular points*, all points m being *regular*. The theorem may be stated as follows:

Through every regular point in the phase space (G^2, P, h, z) *there passes one and only one curve m which is a solution of the system of equations* [11*a*–*c*]. Moreover, by virtue of property [11a], *every such curve is confined to subspace* G^2 = const.

Singular points would occur in our model in downward flow when g and τ_w become antiparallel, or in a channel of varying cross section. Such cases will be dealt with in another publication.

It should not be inferred from the above *local* existence and uniqueness theorem that there are no excluded flow sequences (curves m) for the system shown in figure 1. These will be identified in section 10.

The diagram in figure 5 shows a family of curves m which originate at points M located on one isocline. They are all congruent because they meet the same sequence of vectors θ. Their projection m' onto the thermodynamic plane h, P is a single line m' along which values of z can be entered in the form of a variable parameter. The value of z increases inward along m' reaching a maximum at $z = z^*$ where the trace of δ in (h, P) meets m'.

The diagram of figure 6 displays a family of curves m in Γ_1 which have their maxima at the same elevation $z = z^*$. At $z = 0$ these curves trace a line n_1 which intersects the isentrope $s = s_0(h_0, P_0)$ of the stagnation state at a single point. This proves that for a given value of G there exists a single state 1 in a pipe whose solution peaks at a prescribed value z^* of z and satisfies the condition that $s_1 = s_0$.

Since only solutions with increasing z have physical significance, we conclude that a solution curve m, figure 7, must be regarded as consisting of two branches, m_1 in Γ_1 and m_2 in

Γ_2, both ending at $z = z^*$. Since at $z = z^*$ dh/dz and dP/dz both become infinite some authors object to regarding this particular state as possessing physical significance, but the objection is easily met by excluding the maximum point and treating m_1 and m_2 as open sets. Whereas m_1 is traversed in the positive direction defined by θ, the branch m_2 would have to be traversed in its negative direction. We shall show in section 11 that both possibilities are characterized by an entropy increasing with z and are admissible from the physical point of view.

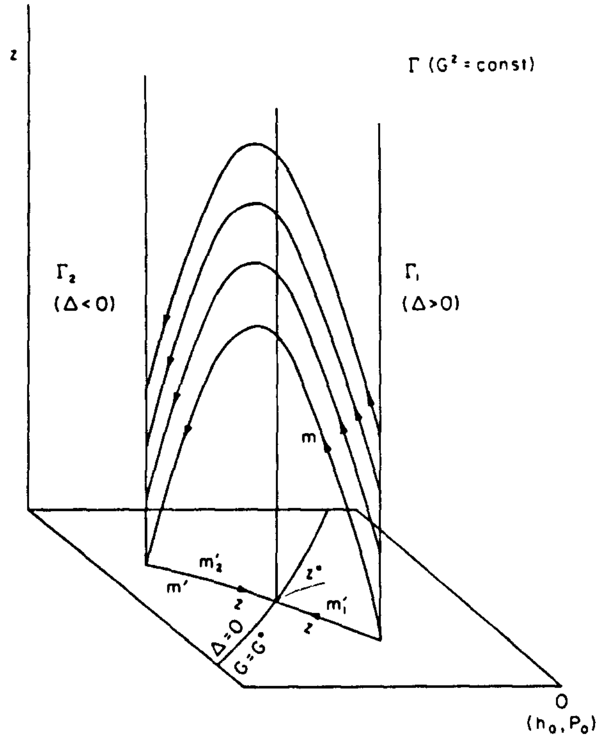

Figure 5. Congruent curves m.

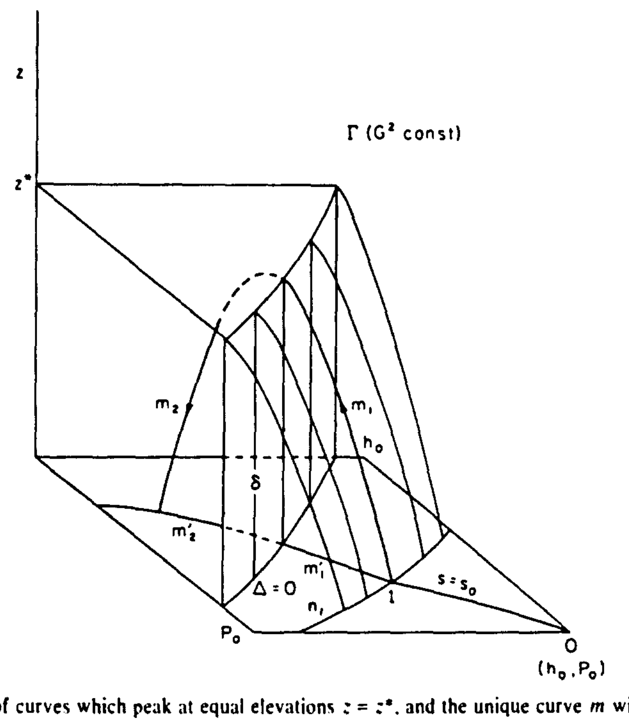

Figure 6. Family of curves which peak at equal elevations $z = z^*$, and the unique curve m with $s = s_0$ at 1.

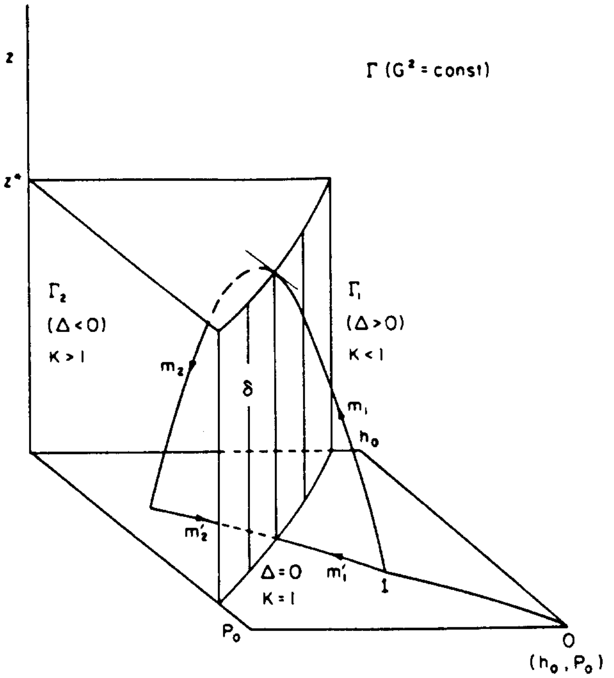

Figure 7. Two branches of curve m.

8. THE VANISHING OF THE DETERMINANT ($\Delta = 0$)

Explicit forms of the relation [21] are listed in table 3. The pair of values P^*, h^* determines a critical value G^* which has the property that it occurs for the maximum value z of z^* characteristic of a given flow rate $G = G^*$. In the present model we discover that

$$G^* = \rho^* a^*, \tag{21a}$$

where ρ^*, a^* are local critical values familiar from elementary gas dynamics.

The general expression for the velocity of propagation of small disturbances

$$a^2 = (\partial P/\partial \rho)_s \tag{23a}$$

can be transformed to

$$a^2 = \rho(\partial h/\partial \rho)_s \tag{23b}$$

or, equivalently, to

$$a^2 = \left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial h}\right)_P + \left(\frac{\partial \rho}{\partial P}\right)_h\right]^{-1} = \frac{\rho}{C_1\rho + C_2}. \tag{23c}$$

Equation [23b] results when we put

$$\left(\frac{\partial P}{\partial \rho}\right)_s = \frac{\partial(P,s)}{\partial(\rho,s)} \quad \text{and} \quad \frac{1}{\rho} = \left(\frac{\partial h}{\partial P}\right)_s = \frac{\partial(h,s)}{\partial(P,s)}.$$

Multiplication of the two Jacobian expressions leads directly to [23b]. The identity in [23c] follows from

$$\left(\frac{\partial \rho}{\partial P}\right)_s = \left(\frac{\partial \rho}{\partial P}\right)_h + \left(\frac{\partial \rho}{\partial h}\right)_P\left(\frac{\partial h}{\partial P}\right)_s \quad \text{where} \quad \left(\frac{\partial h}{\partial P}\right)_s = \frac{1}{\rho}.$$

together with [12a, b]. The application of identity [23c] to the expressions in table 3 leads to the conclusion that [21a] is equivalent to the condition $\Delta = 0$ for $G = G^*$.

The velocity a is completely determined by the thermodynamic state, i.e. by P and h, as is G^*. The former is the velocity of propagation of a plane wave of vanishingly small amplitude. This is the velocity of sound in a single-phase system or the velocity of propagation of a plane disturbance in a two-phase system when the wavelength is large compared to the characteristic dimensions of a bubble or droplet. (The possible dependence of the sonic velocity on frequency cannot come into evidence, because the basic model equations [1]–[3] make no allowance for this phenomenon, or for the associated phenomenon of internal relaxation.)

The cylinder δ which separates Γ_1 from Γ_2 in Γ is thus seen to be the locus of all states for which the given flow rate G becomes the critical flow rate G^*. All states in Γ_1 are thus characterized by the fact that G at them is lower than the local critical value, denoted by G^{**}, whereas in Γ_2 they are larger. This can also be verified with reference to the data in table 3.

The ratio of the actual flow rate, G, to that which would be critical at a given cross section with given values of P, h (denoted earlier by G^{**}), or

$$K = G/G^{**}, \tag{24}$$

can be made to play the same part in this representation as does the Mach number in compressible fluid flow. Evidently

$$K < 1 \quad \text{denotes subcritical conditions,} \tag{25a}$$

$$K = 1 \quad \text{denotes critical conditions,} \tag{25b}$$

$$K > 1 \quad \text{denotes supercritical conditions.} \tag{25c}$$

Along branch m_1 of m, we have

$$K < 1, \Delta > 0 \quad \text{and} \quad G = G^* < G^{**}$$

whereas along branch m_2 we observe

$$K > 1, \Delta < 0 \quad \text{and} \quad G = G^* > G^{**}.$$

These circumstances have been indicated in figure 7.

It may be useful to digress here and to point out that the results in [21a] and [25a–c] are a mathematical consequence of the adopted flow model only through the structure of the matrix A_{ij} in [15].

In general there is no guarantee that the vanishing of $\Delta = |A_{ij}|$ will have the same physical interpretation in alternative theories of two-phase flow as it receives here. Anticipating the results of our physical interpretation of the topological structure of the present vector space θ, we see that choking can occur only at the end of a pipe of given length. When this occurs, the outflow velocity is that of the local velocity of propagation, a^*; it is independent of the gravitational acceleration or of the shearing stress, provided only that τ_w has the form [5]. More generally, this condition is independent of vector B_j in [17]. The tendency to impose this or an equivalent condition, "on physical grounds", on an independently adopted flow model must be resisted, because other models may be inconsistent with such an externally imposed requirement. Alternatively, it is possible to hold the view that only models which are consistent with this condition are acceptable. If this is the case, the conditions under which the determinant vanishes can be used as a discriminant among competing flow models.

9. VARYING THE FLOW RATE G

To complete our understanding of the topological structure of solutions we must now explore the way in which the various characteristic entities move as the flow rate G changes. This we propose to accomplish by observing the relations in the three coupled projections of Γ, namely h, z; P, z; and the thermodynamic diagram h, P.

It should by now be clear that limb m_2 of m corresponds to a sequence of states along which $K > 1$ with $w > a$. This is the familiar supersonic branch of our Fanno line m. The branch m_1 is the subsonic branch. The diagram in figure 6 convinces us that states along m_2 can be reached from the stagnation state P_0, h_0 only if the channel 01 in figure 1 is equipped with a throat. Furthermore, the appearance of supersonic states may lead to the formation of shock waves. In order to avoid such complications (which could, however, have been easily dealt with at the cost of considerably lengthening this paper) we shall confine our discussion to the subsonic branch m_1 whose states can be reached through a gently converging entrance.

The diagram in figure 8 shows the two projections of an isocline which passes through $h = \bar{h}$ and $P = \bar{P}$, together with the directions of θ, for two values of flow rate with

$$G_2 > G_1. \qquad [26a]$$

Reference to table 3 shows that

$$\Delta(G_2) < \Delta(G_1); |N_P(G_2)| > |N_P(G_1)| \quad \text{with} \quad |N_h(G_2)| > |N_h(G_1)|. \qquad [26b]$$

This proves that the slopes of m_1 along a given isocline *increase* with G and that increasing the flow rate forces the solution to reach its maximum in z for a lower value of z^*, so that

$$z^*(G_2) < z^*(G_1). \qquad [26c]$$

A further analysis of the formulae of table 3 proves that Δ vanishes for

$$P^*(G_2) > P^*(G_1) \quad \text{but} \quad h^*(G_2) < h^*(G_1). \qquad [26d]$$

Three solutions which emanate from a given thermodynamic state $M(\bar{h}, \bar{P})$ at z with different flow rates $G_1 < G_2 < G_3$ are shown in figures 9a, b. Their topological relations conform to

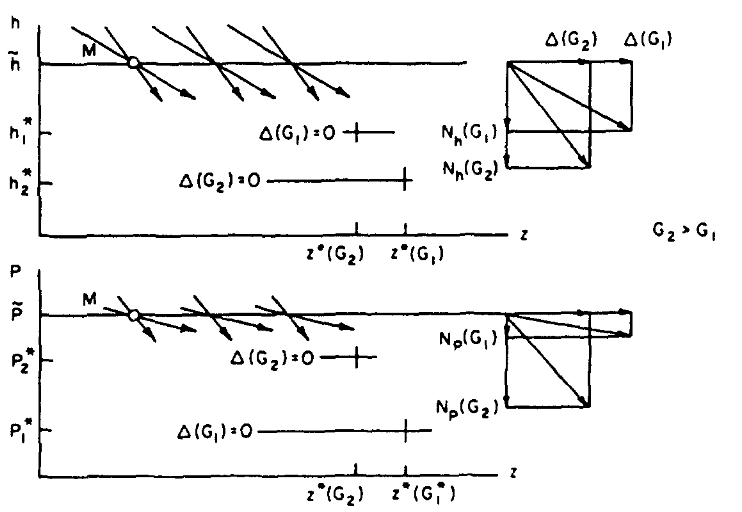

Figure 8. Effect of G on vector field θ.

[26a–d]. The same processes are depicted in the h, P diagram of figure 9c. We have placed here the initial state $M(\bar{h}, \bar{P})$ on the liquid saturation line $x = 0$ which would correspond to the flash horizon in a geothermal well.

As a further aid to the reader's geometric imagination, we introduce figure 10, which illustrates the relative positions of the characteristic curves of constant G^*, a and x (approximately to scale) for water-steam mixtures. We have added segments of curves of ρ = const and s = const indicating the direction in which their values rise.

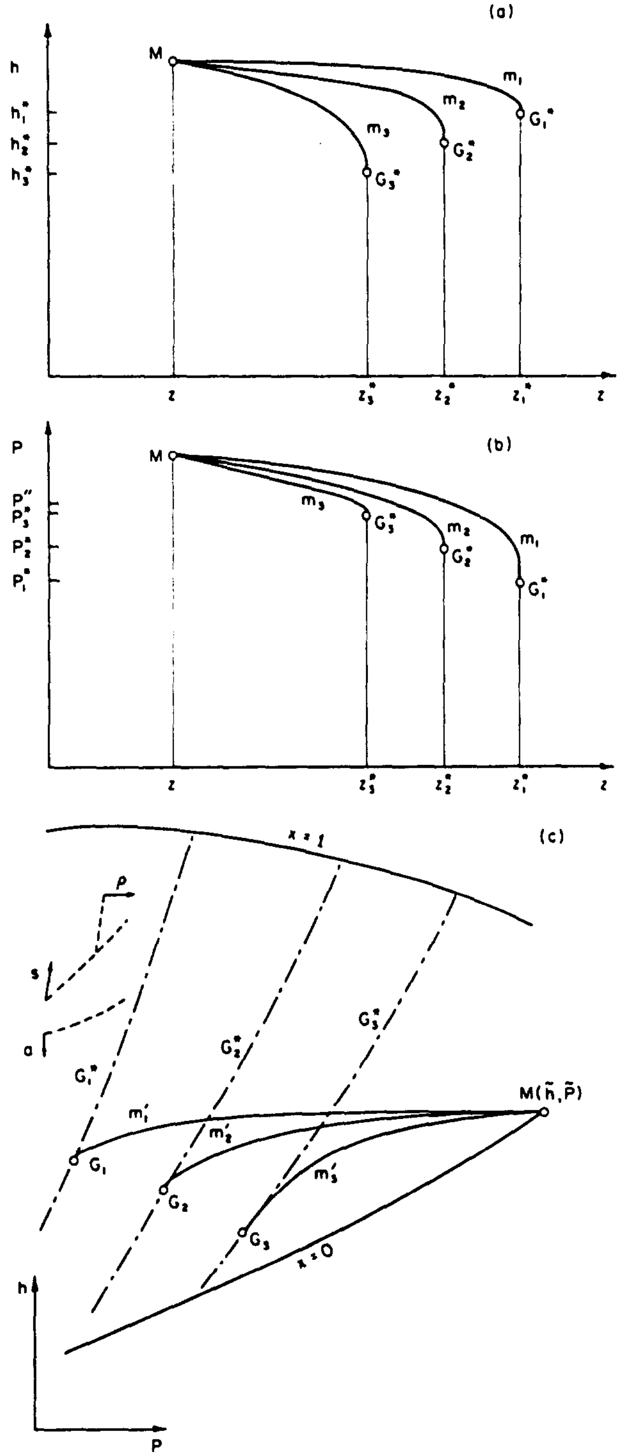

Figure 9. Fanno curves for $G_1 < G_2 < G_3$. (a) Projection h, z, (b) Projection P, z, (c) Projection h, P (topologically correct for two-phase water).

10. PHYSICAL INTERPRETATION. ADMISSIBLE FLOWS AND ADMISSIBLE INITIAL STATES

We revert to the problem depicted in figure 1 and illustrate our reasoning with the aid of the h, P diagram of figure 11, continuing to restrict interest to subcritical flows. We start with an assumed stagnation state $0(h_0, P_0)$. Line 0123 represents the isentrope $s = s_0 = \text{const}$ which is the locus of all states that can exist in cross section 1 at $z = 0$ in figure 1. As the flow accelerates isentropically under the action of a pressure P_a which we imagine decreasing slowly, the consecutive points along $s = \text{const}$ depict increasing values of G.

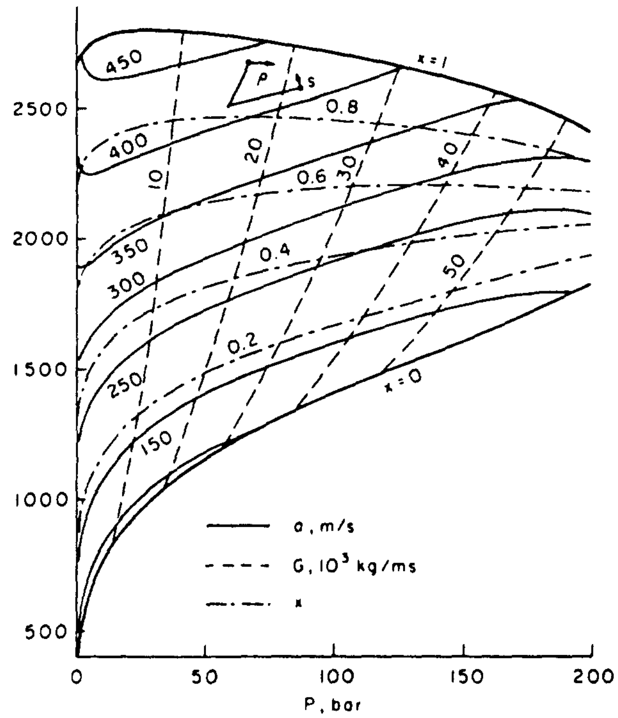

Figure 10. Thermodynamic h, P diagram.

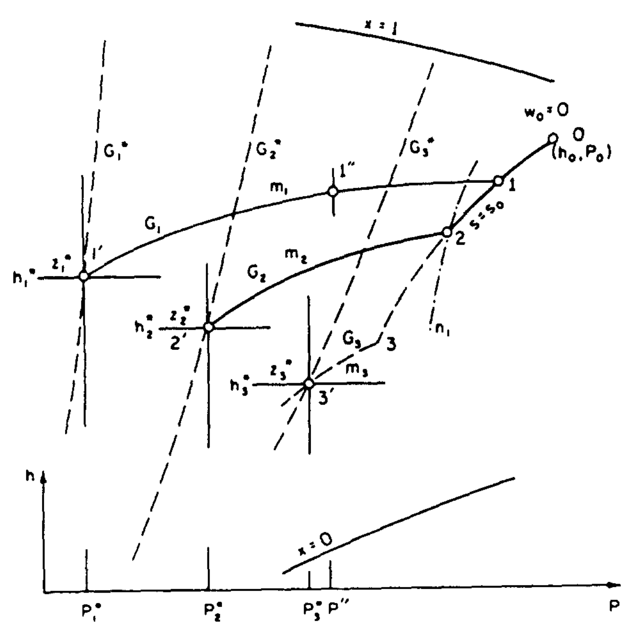

Figure 11. Flow process in two-phase regime.

In the absence of a gravitational field, or for relatively short lengths L, the back-pressure, P_a, at which the exit velocity $w_e = 0$, denoted P_m, is simply $P_m = P_0$. When changes in potential energy are important we must have $P_m < P_0$ in upward flow. If the whole pipe were filled with a liquid of approximately constant density ρ_L, we would put

$$P_m = P_0 - \rho_L g L. \qquad [27]$$

In cases when the liquid is likely to flash at $z < L$, the calculation of P_m must be performed with some care, though the procedure requires no further elaboration.

Each successive curve: m_1, m_2, m_3, peaks at lower and lower values of z^* with $z_1^* > z_2^* > z_3^*$. The corresponding critical pressures are: $P_1^* < P_2^* < P_3^*$. For a sufficiently low value of G, say G_1, the section $z = L$ is attained before solution $G_1 = \text{const}$ intersects the associated critical flow rate at point 1′ where $z_1^* > L$, say at point 1″ with pressure $P_a = P''$. The flow is subcritical from 1 to 1″, the exit pressure $P_e = P''$, and no choking occurs.

As the flow rate is increased, we eventually reach a value, say G_2 at 2, whose end-point $z = L$ exactly coincides with the critical curve $G_2^* = G_2$ with pressure P_2^*. The channel begins to be choked, and the exit pressure $P_e = P_2^*$. If P_a is set to be lower than P_2^*, the flow in the pipe remains unaffected because curve m_2 has peaked at 2′. Flow with a larger value of G, say G_3 at inlet state 3, must peak at $z_3^* < L$ and pressure $P_3^* > P_2^*$. This condition would have to set in inside the pipe and is, therefore, excluded. Indeed, referring back to figure 6; we see that state 2 of figure 11 lies on line n_1. This line, reproduced in figure 11, marks the boundary of all inlet states which are compatible with a subcritical flow regime in a channel of prescribed length L.

It is clear from the diagram and reasoning that flow rate G_2 is the largest attainable from the given stagnation state (h_0, P_0). Thus all flows with $P_a \le P_2^*$ are legitimately described as choked because they occur with a flow rate $G_2 = G_2^* = \text{const}$ and in a regime in which the external pressure $P_a < P_2^*$ has ceased to influence the flow.

The P, z diagram of figure 12 illustrates the various flow sequences: the subcritical flow m_1, the choked flow m_2 and the impossible flow m_3. It follows that in the absence of a convergent-divergent nozzle at pipe inlet, only pressures $P_2 < P < P_m$ can occur at pipe inlet 1 in figure 1, the pressure range $O < P < P_2$ being unattainable in it under the specified constraints.

The diagram in figure 13 displays the pressure distribution in three cases. In figure 13a, which corresponds to flow m_1 of figure 11, the flow is subcritical, $P_e = P_a = P''$, and the fluid leaves the pipe exit smoothly. In figure 13b, the flow corresponds to line 22′ of figure 11. Now the flow is choked with $P_a = P_e = P_2^*$. When $P_a = P_2^*$, the fluid still leaves the pipe exit smoothly, but this is a limiting case. Any further reduction in the backpressure P_a, figure 13c, leaves the flow in the pipe unchanged, and the fluid must undergo a further expansion outside the duct.

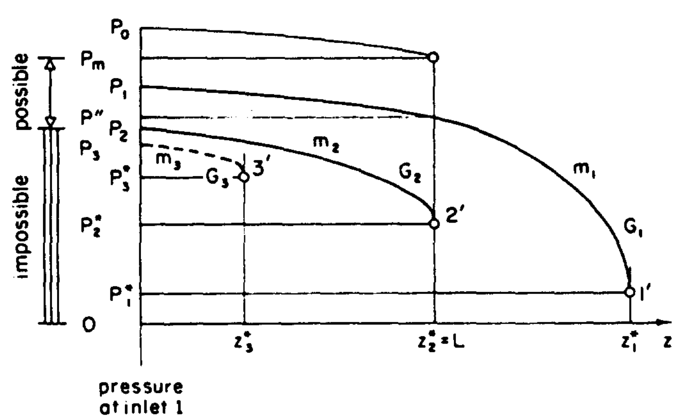

Figure 12. Flow sequences in P, z diagram.

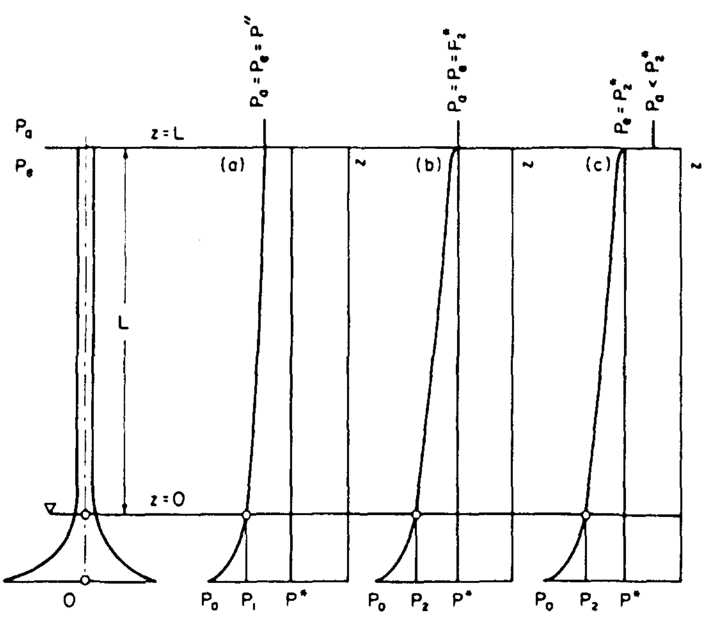

Figure 13. Flow sequences along vertical pipe.

11. ENTROPY

The rate of change of entropy follows from

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{1}{T}\frac{\mathrm{d}h}{\mathrm{d}z} - \frac{1}{\rho T}\frac{\mathrm{d}P}{\mathrm{d}z} \qquad [28a]$$

or

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \frac{4\tau_w}{\rho DT} > 0 \quad (\mathrm{d}s > 0 \text{ implies } \mathrm{d}z > 0). \qquad [28b]$$

This proves, as asserted in section 7, that only upward flows are admissible. Along branch m_2 the entropy would decrease it if followed the direction of the vector θ.

Since the flow is adiabatic [28b] represents the rate of *entropy production* in the flow direction.

12. OTHER THERMODYNAMIC DIAGRAMS

The h, P diagram is not the only one feasible even though its use suggests itself by the simplest choice of dependent functions in [11a–c]. Some readers may prefer the more traditional Mollier (h, s) diagram. The curves of figure 11 have been transferred to such a diagram shown in figure 14; the lettering corresponds to that in figure 11.

There is no difficulty in showing that in the Mollier diagram the choking condition is given by

$$\left(\frac{\partial s}{\partial h}\right)_G = 0. \qquad [29]$$

The reader should now experience no difficulty if he wishes to transfer the characteristic curves to one of the many more feasible thermodynamic diagrams.

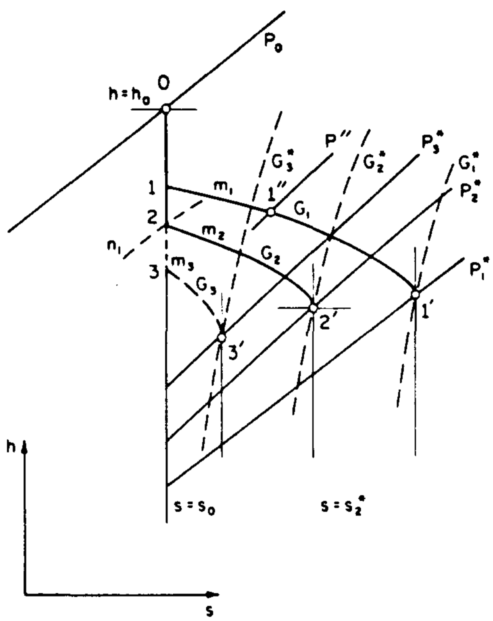

Figure 14. Mollier h, s diagram.

13. CONCLUDING REMARKS ON CHOKING

Since experimentation with choking is relatively easy to perform, we expect that a comparison between the analytic results of this and similar studies and experiments will enable us to make judgements regarding the suitability of individual models for the description of two-phase flows. This justifies the intensity of attention paid to this detail by the authors.

The view is expressed that the condition for choking discovered here, though physically appealing, has no universal physical validity. The true condition must emerge (as it did here) as a joint consequence of

- the flow model
- the equation of state
- the closure equation

not forgetting all the additional assumptions, such as the assumption of thermodynamic equilibrium and of infinite rates of phase change. The shape of the channel and the flow direction in the gravitational field will affect the location of the choked cross section, even though they do not influence the critical flow rate, G^*. In general, it may be observed that the determinant $\Delta = |A_{ij}|$ specifies the thermodynamic *state* at the choked cross-section, whereas Δ and B_i jointly determine the *location* of the choked cross section along the channel.

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